REMARK.—If we designate by B, B_1 the upper and lower bases of a body, and by B_2 the section equidistant from the bases we have

$$x = 0,\ B = a,\ x = h,\ B_1 = a+\beta h+\gamma h^2+\omega h^3,$$

$$x = \tfrac{1}{2}h,\ B_2 = a + \beta\tfrac{1}{2}h + \gamma\tfrac{1}{4}h^2 + \omega\tfrac{1}{8}h^3;\ \text{whence}$$

$$4B_2 = 4a + 2\beta h + \gamma h^2 + \tfrac{1}{2}\omega h^3,\ \text{and consequently}$$

$$B+B_1+4B_2 = 6a + 3\beta h + 2\gamma h^2 + \tfrac{3}{2}\omega h^3,$$

which multiplied by $\frac{1}{6}h$ produces formula (A). We then have

$$V = \tfrac{1}{6}h(B + B_1 + 4B_2),\ \text{a known formula.}$$

For the Sphere.—If in (1) we make $a = 0$, $\beta = 2\pi R$, $\gamma = -\pi$, $\omega = 0$, that function reduces to

$$2\pi Rx - \pi x^2 = \pi x(2R - x),$$

which represents the area of a circle made in a sphere, at the altitude x.

In this case $B = 0$, $B_1 = 0$, $B_2 = \pi R^2$, $h = 2R$; thence

$$V = \tfrac{1}{3}R.4\pi R^2 = \tfrac{4}{3}\pi R^3.$$

For the spherical segment.— Making $B = 0$, $B_1 = \pi h(2R - h)$, $B_2 = \pi.\frac{1}{2}h(2R - \frac{1}{2}h)$, we have

$$V = \pi.\tfrac{1}{6}h^2(2R-h+4R-h) = \tfrac{1}{3}\pi h^2(3R-h).$$

For the cone.—If h is the altitude of a cone, R the radius of the base, the radius made at a height x is $R' = R-(R\div h)x$; whence

$$\pi R'^2 = \pi R^2 - \frac{2\pi R^2}{h}x + \frac{\pi R^2}{h^2}x^2,$$

the expression to which (1) reduces itself when $a = \pi R^2$, $\beta = -2\pi R^2\div h$, $\gamma = \pi R^2\div h^2$, $\omega = 0$. Thence the cone $= \frac{1}{6}h(\pi R^2+\pi R^2) = \frac{1}{3}\pi R^2h$.

In the same way we obtain for the volume of the frustum of a cone

$$V = \tfrac{1}{6}h[\pi R^2+\pi R^2+\pi(r+R)^2] = \tfrac{1}{3}\pi h(R^2+r^2+Rr).$$

NOTE ON BILINEAR TANGENTIAL COORDINATES.

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TANGENTIAL equations are usually written as homogeneous, and so analogous to trilinear equations; but interesting relations to the ordinary Cartesian coordinates are thus unnoticed. If the general equation of the first degree in two variables be divided through by its absolute term, and signs changed, it takes the form

$$px + qy - 1 = 0.$$

In the Cartesian system the two coefficients of this equation, p and q, are the reciprocals of the intercepts of its locus upon the two axes; and a recip-

rocal tangential interpretation will be put upon it if the same meanings be assigned to x and y, when p and q will take the significations of the Cartesian variables, viz., the distance of the point (the envelope denoted by the equation) from either axis in directions parallel in each case to the other axis. We are thus led to the system of bilinear tangential coordinates, wherein the two coordinates of a line are defined as the reciprocals of its intercepts on two given axes.

The origin of this system, being the envelope of all lines whose coordinates are infinite, takes the place which in the Cartesian system is occupied by the locus at infinity; and by the aid of this analogy it becomes possible to write the reciprocals of theorems involving measurement, as readily as those involving descriptive relations only.

For instance the equation of condition

$$AB' = A'B$$

which connects the coefficients of the Cartesian equations of parallels, will hold for the coefficients of the equations of points, if the latter are in a line with the origin. Points thus situated may therefore be spoken of as parallel to one another.

All formulæ relating to the Cartesian equations of right lines, which involve the angle ω of the axes, may be transcribed without the change of a letter as analogous formulæ for the equations of points, provided that ω in the tangential formulæ be defined to mean the angle between the positive direction of one axis and the negative direction of the other, and so to be the supplement of the angle denoted by the same letter in the Cartesian system. Thus two points are *perpendicular* to each other (their directions from the origin are at right angles), in case the coefficients of their equations fulfill the condition

$$AA'+BB'-(AB'+A'B)\cos\omega = 0.$$

In like manner, the formula for the tangent of the angle between the lines drawn from the origin to two points whose equations are given is the same as that which denotes the angle between two lines given by their Cartesian equations.

The quantity $[(x''-x')^2 + 2(x''-x')(y''-y')\cos\omega + (y''-y')^2]^{\frac{1}{2}}$, the familiar formula for the distance between two points, is found, when x', y', x'', y'' are the coordinates of lines, to denote a function of the position of the two lines in relation to the origin as well as to each other, which function, for analogy's sake, may be called the *punctual distance* of the two lines. It may be defined as the product of two ratios; first, that of the difference of the intercepts of the lines on an axis to the product of these intercepts; second, the ratio of the distance of the intersection of the lines from the

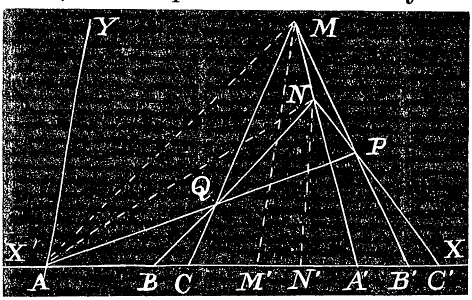
origin, to the distance of the same point (measured in a direction parallel to the other axis) from the axis formerly employed.

As an example of the reciprocation of metrical theorems, the following may be given; in which the left-hand theorem is the elementary proposition that the line parallel to the base of a triangle divides the sides proportionally.

If a line be drawn parallel to the base of a triangle, so as to cut the other sides, the ratio in which the linear distance on either side, between the vertex and the extremity of the base, is divided by the intersection of the parallel line, is equal to the ratio in which the other side is divided.

If a point be taken parallel to the vertex of a triangle, and joined to the extremities of the base, the ratio in which the punctual distance at either of these extremities, between the base and the adjacent side, is divided by the line drawn to the parallel point, is equal to the ratio in which the like distance is divided at the other ext'y.

In the subjoined figure, AX and AY are the axes, PMN the triangle, and Q the point parallel to P; the lines joining these four points two and two are produced to meet the axis of X, and the points M and N are joined both to the origin at A and (by lines parallel to AY) to M' and N' on the axis of X.



By writing out in full the values of the "punctual distances" referred to in the theorem, the proportion between them is put in the form:

$$\frac{A'B}{AA'.AB}\cdot\frac{AN}{N'N} : \frac{BC'}{AB.AC'}\cdot\frac{AN}{N'N} = \frac{A'C}{AA'.AC}\cdot\frac{AM}{M'M} : \frac{CB'}{AC.AB'}\cdot\frac{AM}{M'M}.$$

By cancellation of common factors this becomes:

$$A'B : \frac{BC'}{AC'} = A'C : \frac{CB'}{AB'};$$

which may be written

$$\frac{BA'}{A'C}\cdot\frac{CB'}{B'A}\cdot\frac{AC'}{C'B} = -1,$$

that is, the six points A, B, C, A', B', C' are in involution.

Since the theorem imposed no restriction upon the position of the triangle MNP relatively to the axes, it is apparent that, as a reciprocal to the proposition that a parallel to the base of a triangle divides the sides proportionally, we have proved that

"The three points in which any line cuts the sides of a triangle, and the projections, from any point in the plane, of the vertices of the triangle on the same line are six points in involution."

The fact that this proposition has thus been derived from another as its reciprocal, will not at all interfere with the deduction in the ordinary manner of the reciprocal theorem concerning six rays in involution joining any point to the vertices of a complete quadrilateral.

On the other hand, as no use whatever has been made of harmonic properties to obtain the theorem, these may all be deduced as special applications, by drawing the transversal AX through two of the points in which sides of the triangle MNP meet the lines joining Q to the opposite vertices, thus making these points foci of the involution.

Or, should it be preferred to proceed more nearly in the usual course, a definition of the punctual distance, equivalent to the one already stated, may be given in terms of angular functions. Thus, in the figure, the punctual distance between NC' and NB is

$$\frac{\sin NAX \sin BNC'}{N'N \sin ANB \sin ANC'};$$

and it is easily shown that the ratio in which the punctual distance of two out of three convergents is divided by the third is equal to the anharmonic ratio of the pencil formed by adding to the three convergents a line which joins the point of convergence to the origin.

A NEW AND USEFUL FORMULA FOR INTEGRATING CERTAIN DIFFERENTIALS.

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PROBLEM.—*To integrate* $v^n du$ *in terms of the descending powers of* v.

Let us assume

$$\int v^n du = yv^n + y_1 v^{n-1} + y_2 v^{n-2} + \ldots + y_t v^{n-t} + \int z v^{n-t-1} dv. \qquad (1)$$

Differentiating (1), and arranging with reference to v, we have

$$0 = \left.\begin{matrix}-du\\ dy\end{matrix}\right| v^n + \left.\begin{matrix}nydv\\ dy_1\end{matrix}\right| v^{n-1} + \left.\begin{matrix}(n-1)y_1 dv\\ dy_2\end{matrix}\right| v^{n-2} + \ldots + \left.\begin{matrix}(n-t)y_t dv\\ zdv\end{matrix}\right| v^{n-t-1} \qquad (2)$$

Now since (2) is true for every value of v, according to the principle of indeterminate coefficients, we have